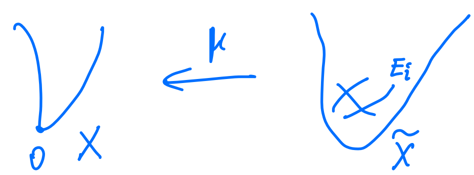
Minimal log discrepancy and orbifold curves

(Joint with Zhengyi Zhou)

· $\mathrm{mld}(0, X)$ X : $\mathbb{Q}$-Gorenstein klt sing. isolated sing.



$$K_{\tilde X} + \sum_i E_i = \mu^* K_X + \sum_i A(E_i) E_i.$$

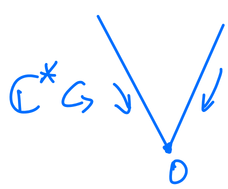
klt $\Rightarrow A(E_i) > 0, \ \forall i.$

$$\mathrm{mld}(0, X) = \min_i \{ A(E_i) \}$$

Dolgachev

Pinkham, Demazure, Kollár.

Fano cone sing. : good $\mathbb{C}^*$-action. (Sasaki Geometry)

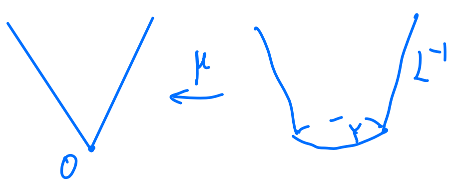


$$X /\!/ \mathbb{C}^* = Y \quad \text{orbifold (Deligne-Mumford stack)}$$

$$X /\!/ \mathbb{C}^* = (X - 0)/\mathbb{C}^*$$

Ex: If $\mathbb{C}^*$ action is free on $X^n \setminus 0$, then Y is smooth proj. mfd.

klt $\Leftrightarrow$ $\underbrace{Y^{n-1} \text{ is Fano}}_{-K_Y > 0}$, $X = C(Y, L)$, $-K_Y = r L$, $r > 0$.



$$+K_{\tilde X} + Y|_Y = K_Y = -r \cdot L$$

$$(\mu^* K_X + A \cdot Y)|_Y = A \cdot [Y]|_Y$$

$$\Rightarrow A(Y) = r = \mathrm{mld}(0, X)$$

Fact: Y is Fano mfd., $-K_Y = r \cdot L$, $r > 0$. Then $r \leq \dim Y + 1 = n$

"=" $\Leftrightarrow Y \cong \mathbb{P}^{n-1}$, $L = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$.

(Kobayashi-Ochiai)

$$H^i(Y, kL) = H^i(Y, K_Y \underbrace{-K_Y}_{= rL} + kL) = H^i(Y, \underbrace{K_Y + (k+r)L}).$$

L ample $\Rightarrow$ $H^i(Y, kL) = \begin{cases} 0, & \text{if } i=0,\ k<0 \\ 0, & \text{if } i>0,\ k+r>0 \Leftrightarrow k>-r. \end{cases}$

$\Rightarrow$ $H^i(Y, kL) = 0$, $\forall\, i$, if $k = -r+1, \cdots, -1$.

$\Rightarrow$ $\underbrace{\chi(Y, kL) = 0}_{= P_{\dim Y}(k)}$ $\quad k \in \{\underbrace{-r+1, \cdots, -1}_{r-1}\}$ $\Rightarrow$ $r-1 \le \dim Y$

$\Downarrow$

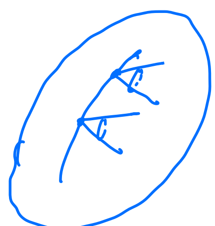
$r \le \dim Y + 1.$

$\rightsquigarrow$ Shokurov's Conj.: $\underset{\substack{\| \\ \dim Y+1}}{\mathrm{mld}(o, X) \le \dim X}$ "=" iff X is smooth.

Result:

- $\dim X \le 3$, Kawamata, Markushevich.
- Toric (Borisov), T-variety of complexity 1 (Meier)
- Quotient sing. (Reid-Tai), Nakamura-Shibata, ...
- Complete intersection (Ein-Mustata).

- Quasi-regular Fano cone. n-dim A_{k-1}-sing



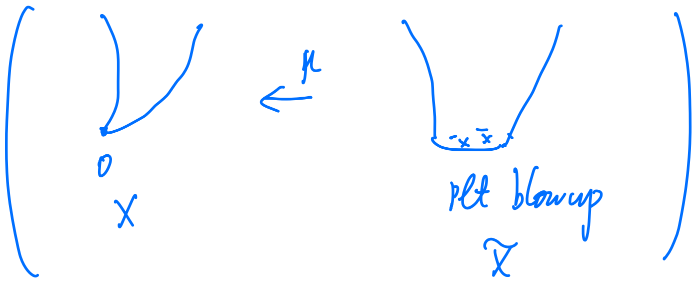
Ex: $\{z_1^2 + z_2^2 + \cdots + z_n^2 + z_{n+1}^k = 0\} \subset \mathbb{C}^{n+1}$

$\mathbb{C}^* \curvearrowright (k, k, \cdots, k, 2)$

$$\underset{\substack{\| \\ (X-o)/\mathbb{C}^*}}{X/\!/\mathbb{C}^*} = Y) = (Y, \Delta) = \begin{cases} \underline{(Q^{n-1}, (1-\frac{1}{m})\Delta)} & k=\text{even} \\ (\mathbb{P}^{n-1}, \underline{(1-\frac{1}{2m+1})\Delta}) & k=\text{odd} \end{cases}$$

Thm (L.-Zhou) For isolate Fano cone sing. $\mathrm{mld}(o,X)\le \dim X$.

$\mathbb{C}^*$



- A Formula for mld

$(K_{\tilde{X}}+Y)|_Y = K_Y + \Delta$

$K_Y = -r\cdot L$

$(\tilde{X}, Y)$ plt.

$\mathbb{C}^n/\frac{1}{m}(a_1,\cdots,a_n)$, $Y=\{z_1=0\}$

$X = C(Y, L)$

(Kollár: Seifert $\mathbb{G}_m$-bundle)

$e \ne g \in \mathbb{Z}_m \curvearrowright \mathbb{C}^n$

$g\cdot z_i = e^{\frac{2\pi\sqrt{-1}}{m}\cdot w_i(g)}$ $\quad 1\le w_i(g)<m$

Prop (L.-Zhou)

$$\mathrm{mld}(o,X) = \min_{p,\ g\in \mathrm{Stab}(p),\ g\ne e}\left\{ r,\ \frac{1}{m}\Big(r\cdot w_1(g)+\sum_{i=2}^{n} w_i(g)\Big)\right\}$$

p: strata of orbifold locus

Rmk: Quotient sing. $\mathbb{C}^n/G$

Reid-Tai

$\mathrm{mld}(o,X) = \{\mathrm{age}(g) : g\ne e\in G\}$

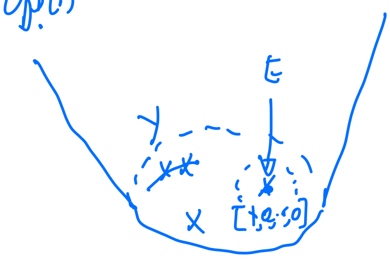
$g\cdot z_i = e^{\frac{2\pi\sqrt{-1}}{m}\cdot w_i(g)}$

$\mathrm{age}(g) = \frac{1}{m}\sum_i w_i(g)$.

Ex: $\mathbb{C}^* \curvearrowright \mathbb{C}^n$, $\quad Y = (\mathbb{C}^n - 0)/\mathbb{C}^* = \mathbb{P}(a_1, a_2, \dots, a_n)$.

$t\cdot(z_1,\dots,z_n) = (t^{a_1}z_1, \dots, t^{a_n}z_n) \qquad a_1 \geq a_2 \geq \dots \geq a_n > 0.$

$r\underset{\substack{\uparrow \\ \mathcal{O}_P(1)}}{L} := -K_Y$: $\quad \underline{r = a_1 + a_2 + \dots + a_n} > \mathrm{mld}(\mathbb{C}^n, 0) = n$.



$p = [1, 0, \dots, 0]$. $\quad \mathrm{Stab}(p) = \mathbb{Z}_{a_1} = \langle g_1 \rangle$

$w_1(g_1) = 1, \quad w_i(g) = a_1 - a_i, \quad 2 \leq i \leq n.$

$$\frac{1}{a_1}\Big(\underset{\substack{\shortparallel \\ a_1+\dots+a_n}}{r}\cdot 1 + \sum_{i=2}^{n}(a_1 - a_i)\Big) = \frac{n\cdot a_1}{a_1} = n = \mathrm{mld}(\mathbb{C}^n, 0)$$

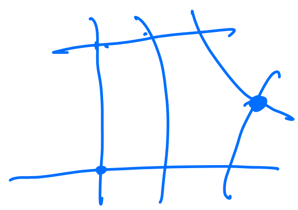
• Mori: Y is Fano mfd. $\Rightarrow \exists$ (rational) curve C s.t. $-K_Y \cdot C \leq \overset{\dim Y+1}{\overset{\shortparallel}{n}}$

$f: \mathbb{P}^1 \to Y$.

$$\dim \mathrm{Mor}_{[f]}(\mathbb{P}^1, Y; f|_{\{0,\infty\}}) \geq -K_Y \cdot f_*(\mathbb{P}^1) - \dim Y.$$

$\mathbb{C}^*$

If $> 1 \Rightarrow$ bend-and-break



$$\underset{\substack{\shortparallel \\ r(L\cdot C)}}{-K_Y \cdot C} \leq \dim Y + 1 \Rightarrow \underline{r \leq \frac{\dim Y+1}{L\cdot C} \leq \dim Y + 1}$$

Prop (L.-Zhou)

$$\mathrm{mld}(0,X) \leq \min\left\{ -K_Y \cdot f_* C + \mathrm{age}(g^{-1}),\ f \in \mathrm{Mor}_{(1,g)}(\mathbb{P}^1(1,\ell), Y),\ f|_{\{0,\infty\}},\ \begin{matrix} f(0)\in Y_1 \\ f(\infty)\in Y/g \end{matrix} \right\}$$

$g \in \mathrm{Stab}(p)$

$$\leq$$

$$\dim \mathrm{Mor}_{(1,g)}(\mathbb{P}^1(1,\ell), Y) + \dim Y$$

$$\leq$$

$$1 + \dim Y$$

Abramovich-Graber-Vistoli

twisted map

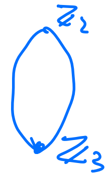
$\updownarrow$

good orbifold map

Chen-Ruan

bend and break in orbifold setting $\longrightarrow$

Chen-Tseng

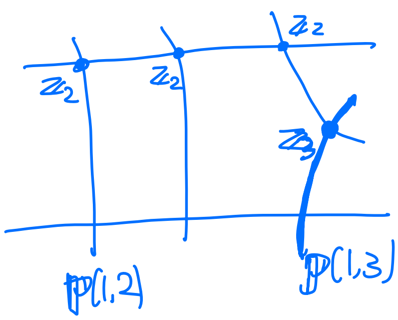


Ex: $\mathbb{C}^* \curvearrowright \mathbb{C}^2_{(2,3)} \rightsquigarrow Y = \mathbb{P}(2,3)$

$\mathbb{P}(1,2) \longrightarrow \mathbb{P}(2,3)$

$[u_1, u_2] \longmapsto [u_2, u_1^3]$

$$-K_Y \cdot f_* C = 5\underbrace{(H\cdot\mathbb{P}(1,2))}_{\frac{1}{2}} + \frac{1}{2} = \frac{6}{2} = 3 > 2$$



$$5\cdot\frac{1}{3} + \frac{1}{3} = \frac{6}{3} = 2 \ \checkmark$$